*The Ten Nodes of the Rational Sextic and of the Cayley Symmetroid.**

BY ARTHUR B. COBLE.†

Introduction.

The general rational plane sextic with ten nodes occupies a unique position among all rational plane curves in that it is the rational curve of lowest order which can not be transformed by ternary Cremona transformation into a straight line, that is to say its order can not be reduced by such transformation. It may, however, be transformed into other rational sextics, and this can be accomplished by Cremona transformations of infinitely many distinct types. One of the principal results of this paper is that the sextic and all of its sextic transforms are comprised under precisely $2^{13}.31.51$ projectively distinct types.

The intimate relation between the ten nodes of a rational plane sextic and the ten nodes of that quartic surface known as the Cayley symmetroid has been pointed out by J. R. Conner. ‡ It is not surprising therefore to find that a similar fact is true of the symmetroid under regular Cremona transformation in space.

The methods of investigation here employed have been set forth in an earlier series of papers by the writer. § Some of the points of view may be recapitulated briefly as follows. We shall be interested in a Cremona transformation C only in so far as it disturbs projective relations so that for our purposes $C \equiv \pi C \pi'$ where π, π' are arbitrary projectivities. If C has the singular points, or F-points, $p_1, \ldots, p_\rho$, and C^{-1} the F-points $q_1, \ldots, q_\rho$ then C transforms curves of order x_0 and multiplicities x_i at p_i into curves of order x'_0 and multiplicities x'_j at q_j $(i, j=1, \ldots, \rho)$ where x' is determined in terms of x by the linear transformation, $L(C)$,

$$(1) \qquad L(C): \; x'_0 = m x_0 - \Sigma_{i=1}^{i=\rho} r_i x_i, \quad x'_j = s_j x_0 - \Sigma_{i=1}^{i=\rho} \alpha_{ij} x_i .$$

* Read by title at the meeting of the Chicago Section of the American Mathematical Society, April, 1919.

† This investigation has been carried on under the auspices of the Carnegie Institution of Washington, D. C.

‡ "The Rational Sextic Curve and the Cayley Symmetroid," this Journal, Vol. XXXVII (1915), p. 29.

§ "Point Sets and Cremona Groups," Part II, *Trans. Amer. Math. Soc.*, Vol. XVII (1916), p. 345; referred to hereafter as P. S. II.

In (1) the coefficients are m, the order of C; r_i, the order of F-point p_i; s_j, the order of the F-point q_j; and α_{ij}, the number of times the fundamental curve, or F-curve, of p_i passes through q_j.

The product CC' of two Cremona transformations can be unique only when the position of the F-points p'_i of C' with respect to the F-points q_j of C^{-1} is definitely specified. In order to limit the possibilities which arise in this connection we require that the points p_i shall be in a set of n points P_n^2 and the points q_j in a set of n points Q_n^2 such that the further pairs $p_{\rho+1}$, $q_{\rho+1}, \ldots, p_n, q_n$ are pairs of ordinary corresponding points of C. This amplifies the linear transformation $L(C)$ by the equations

$$(2) \qquad x'_l = -(-1)x_l \qquad (l=\rho+1, \ldots, n),$$

and the two sets P_n^2, Q_n^2 are called *congruent under* C. In forming the product CC' we require that the points of $P_n'^2$ shall coincide *in some order* with the points of Q_n^2. This possibility of reordering the points of a set—a non-projective operation for $n>4$—is accounted for by adjoining to the linear transformations $L(C)$ those additional ones constituting a $g_{n!}$ which permute the variables $x_1, \ldots, x_n$. Thus the operations involved in passing from a set P_n^2 to all sets Q_n^2 congruent in some order to P_n^2—operations which constitute a group $G_{n,2}$—are reflected by simple isomorphism in the transformations $L(C)$ of the group $g_{n,2}$ generated by $g_{n!}$ and the transformation $L(C)$ determined by a single quadratic transformation C, since the general Cremona transformation is a product of properly ordered quadratic transformations. Obviously when a set P_n^2 is in question this general transformation is restricted to have $\rho \overline{<} n$ F-points.*

We are concerned here with the set P_{10}^2 of the nodes of a rational plane sextic and can state at once the theorem

(3) *A sextic S with nodes P_{10}^2 can be transformed into a sextic $\bar{S}$ with nodes Q_{10}^2 by ternary Cremona transformation if and only if the sets P_{10}^2 and Q_{10}^2 are congruent.*

For if S is transformed by C into $\bar{S}$ the ρ F-points of C must be all within P_{10}^2 else the order of the transform is greater than 6. Hence $\rho \overline{<} 10$. If $\rho < 10$ the nodes of S in P_{10}^2 which are ordinary points of C pass into nodes of $\bar{S}$ in the congruent set Q_{10}^2.

The arithmetic group $g_{n,2}$ simply isomorphic with $G_{n,2}$ has integer coefficients. We shall prove in § 1 that there is only a finite number of projectively distinct sets Q_{10}^2 congruent to the set P_{10}^2 *when P_{10}^2 is the set of nodes of S*, and that, for all the operations of $G_{10,2}$ whose isomorphic elements in

* These remarks are amplified in P. S. II, § 1.

$g_{10,2}$ have coefficients congruent modulo 2 to those of the identity, the set Q^2_{10} is projective to P^2_{10} and therefore may be made to coincide with P^2_{10} by a subsequent projectivity. These elements form an invariant subgroup $\bar{g}_{10,2}$ of $g_{10,2}$ whose factor group $g^{(2)}_{10,2}$* is finite and of order $10!2^{13}.31.51$.

An important problem is now apparent. Since $g_{10,2}$ is infinite and discontinuous (P. S. II, § 4 (18)) and $\bar{g}_{10,2}$ is of finite index under $g_{10,2}$ there follows that an infinite discontinuous Cremona group $\bar{G}_{10,2}$ exists which transforms the sextic S into itself. $\bar{G}_{10,2}$ also will contain an invariant subgroup $\bar{\bar{G}}_{10,2}$ which consists of those elements of $\bar{G}_{10,2}$ for which every point of S is fixed. It may be and probably is true that $\bar{\bar{G}}_{10,2}$ is merely the identical transformation, but in any case the factor group of $\bar{\bar{G}}_{10,2}$ under $\bar{G}_{10,2}$ will be represented by a discontinuous group of elements of the form

$$t' = \frac{at+b}{ct+d},$$

where t is the parameter on the rational curve S. From certain geometrical considerations it seems reasonable to think that this discontinuous group is of genus 4, and that the ten nodes of S can be expressed by means of Riemannian modular functions of genus 4.

The ten nodes P^3_{10} of the Cayley symmetroid Σ, discussed in Part II, behave under *regular*† Cremona transformations in space much like the ten nodes of S under ternary transformation. One novelty introduced in § 4 is the *dilation* of the regular group in a space S_k into a subgroup of the regular group in a higher space S_{k+l}.

PART I.

The Ten Nodes P^2_{10} of the Sextic S.

§ 1. *The Equivalence of the f-curves of P^2_{10} under $\bar{G}_{10,2}$.*

The first theorem which we shall use is

(4) *The group $\bar{G}_{10,2}$ which leaves the sextic S unaltered is generated by the involutions conjugate under $G_{10,2}$ to the Bertini involution.*

We recall that the Bertini involution is defined as follows. Given eight points $p_1, \ldots, p_8$ in the plane, the ∞^3 sextics with nodes at these points have the property that the ∞^2 sextics of the system on a point x pass also through another point y, the copoint of x in the involution B. Obviously every sextic

* The factor groups $g^{(2)}_{n,k}$ for the group $g_{n,k}$ have been identified with known groups in the author's paper entitled "Theta Modular Groups Determined by Point Sets," this Journal, Vol. XL (1918), p. 317; cited hereafter as T. M. Groups. This paper emphasizes the geometric possibilities of the particular cases $g_{2p+2,p}$. It is of interest to find that other cases also have geometric applications.

† Cf. P. S. II, § 4, or § 4 of this paper for the definition.

of the system is a fixed curve, and every additional node of such a sextic is a fixed point of the involution whence it leaves the sextic S with nodes at $p_1, \ldots, p_8, p_9, p_{10}$ unaltered. By permutation of the points of P_{10}^2 all the $\binom{10}{8}$ Bertini involutions attached to the set P_{10}^2 are obtained. Moreover, if C is any Cremona transformation with F-points at P_{10}^2, then CBC^{-1} also leaves S unaltered. For C transforms S into a sextic S' with nodes at Q_{10}^2, B leaves S' unaltered, and C^{-1} transforms S' back into S. Hence the conjugate set of involutions described in (4) all belong to $\overline{G}_{10,2}$. The proof that they generate $\overline{G}_{10,2}$ will appear later. Meanwhile two objects conjugate under $\overline{G}_{10,2}$ will be called equivalent, and this relation of equivalence will be denoted by the symbol $\equiv$.

The f-curves of the set P_{10}^2 are the transforms by Cremona transformation of the sets of directions about the points. Instead of the general Cremona transformation we may make repeated use of the quadratic transformation $A_{j_1j_2j_3}$ with F-points at $p_{j_1}, p_{j_2}, p_{j_3}$. Beginning then with the set of directions about p_1, it becomes under the $g_{n!}$ of permutations of the points, a set of directions about any one of the ten points. Applying A_{123} to the set of directions at p_1 it becomes the line on q_2q_3, and under $g_{n!}$ this becomes any line q_iq_j. Applying A_{123} to the line p_4p_5 it becomes a conic on $q_1q_2q_3q_4q_5$. Proceeding in this way the totality of f-curves of the set P_{10}^2 is obtained. We shall denote by its *signature*, $f_r(j_1^{k_1}, j_2^{k_2}, \ldots, j_{10}^{k_{10}})$, an f-curve of order r with multiple points of orders $k_1, \ldots, k_{10}$ at the points $p_1, \ldots, p_{10}$, respectively. A systematic derivation of the types of f-curves is carried out in the following table (5):

	f-curve	operated upon by	becomes	which is	
	$f_0(1)$	A_{123}	$f_1(23)$*		
		A_{234}	$f_0(1)$		
	$f_1(23)$	A_{123}	$f_0(1)$		
		A_{124}	$f_1(23)$		
		A_{145}	$f_2(12345)$*		
	$f_2(12345)$	A_{345}	$f_1(12)$		
		A_{456}	$f_2(12345)$		
		A_{567}	$f_3(12345^267)$*		
(5)		A_{678}	$f_4(123456^27^28^2)$	$\equiv f_2(12345)$	(1^0)
	$f_3(12345^267)$	A_{567}	$f_2(12345)$		
		A_{467}	$f_3(12345^267)$		
		A_{578}	$f_3(12345^267)$		
		A_{678}	$f_4(12345^26^27^28)$	$\equiv f_2(12348)$	(1^0)
		A_{589}	$f_4(12345^36789)$*		
		A_{789}	$f_5(12345^267^38^29^2)$	$\equiv f_3(12345^267)$	(2^0)
		A_{8910}	$f_6(12345^2678^39^310^3)$	$\equiv f_4(1234678^3910)$	(3^0)
	$f_4(12345^36789)$	A_{567}	$f_3(12345^289)$		
		A_{678}	$f_5(12345^36^27^28^29)$	$\equiv f_3(123456^29)$	(2^0)
		A_{5610}	$f_4(12345^36789)$		
		A_{6710}	$f_6(12345^36^37^38910^2)$	$\equiv f_4(12345^36789)$	(3^0)

New types of f-curves as they are obtained are starred, and these new types are in turn subjected to transformation. However, as the process goes on, the new types obtained are equivalent under $\overline{G}_{10,2}$ to earlier types and these need not be transformed afresh.

In order to prove the equivalences (1^0), (2^0), (3^0) listed in the table (5), and at the same time to verify that the two further equivalences

(4^0) $$f_3(i_1 i_2 i_3 i_4 i_5 i_6 j^2) \equiv f_3(i_1 i_2 i_3 i_4 i_5 i_6 k^2),$$

(5^0) $$f_4(i_1 i_2 i_3 i_4 i_5 i_6 i_7 j k^3) \equiv f_4(i_1 i_2 i_3 i_4 i_5 i_6 i_7 j^3 k),$$

are valid we begin with the equivalence,

(6) $$f_0(i) \equiv f_6(i^3 j_1^2 j_2^2 j_3^2 j_4^2 j_5^2 j_6^2 j_7^2)$$

which is derived at once from a Bertini involution. If the two members of this equivalence be transformed by C then the two transforms are themselves equivalent under the transform of the Bertini involution by C, whence according to (4) they are equivalent under $\overline{G}_{10,2}$. Transforming (6) by $A_{ij_1j_2}$, $A_{ij_3j_4}$, $A_{ij_5j_6}$ and $A_{ij_7j_8}$ successively we get

(7) $$f_1(j_1 j_2) \equiv f_5(i^2 j_1 j_2 j_3^2 j_4^2 j_5^2 j_6^2 j_7^2),$$

(1^0) $$f_2(i j_1 j_2 j_3 j_4) \equiv f_4(i j_1 j_2 j_3 j_4 j_5^2 j_6^2 j_7^2),$$

(4^0) $$f_3(i^2 j_1 j_2 j_3 j_4 j_5 j_6) \equiv f_3(j_1 j_2 j_3 j_4 j_5 j_6 j_7^2),$$

(5^0) $$f_4(i^3 j_1 j_2 j_3 j_4 j_5 j_6 j_7 j_8) \equiv f_4(i j_1 j_2 j_3 j_4 j_5 j_6 j_7^3 j_8).$$

If now we transform (4^0) by $A_{j_1j_7j_8}$ and $A_{j_7j_8j_9}$ we get

(2^0) $$f_5(i^2 j_1^3 j_2 j_3 j_4 j_5 j_6 j_7^2 j_8^2) \equiv f_3(j_1 j_2 j_3 j_4 j_5 j_6 j_7^2),$$

(3^0) $$f_6(i^2 j_1 j_2 j_3 j_4 j_5 j_6 j_7^3 j_8^3 j_9^3) \equiv f_4(j_1 j_2 j_3 j_4 j_5 j_6 j_7^3 j_8 j_9),$$

whence all the equivalences used in limiting the table (5) have been established.

A glance at the list of equivalences established shows that the signatures of equivalent f-curves are congruent modulo 2, and further that no two of the non-equivalent f-curves in the first column of table (5) have signatures which are congruent modulo 2. This is to be expected since the signatures, $f_{r_i}(p_1^{a_{i1}}, p_2^{a_{i2}}, \ldots, p_{10}^{a_{i10}})$, of the f-curves of P_{10}^2 arise from the columns other than the first of the matrices of the linear transformations L of (1) and (2), and the transformation L which correspond to the Bertini involutions (and therefore also to the conjugates of the Bertini involutions) are congruent to the identity modulo 2. Thus we have proved that

(8) *Under the group generated by the conjugate set of involutions which contains a Bertini involution, the infinite number of f-curves of P_{10}^2 divide into* $527 = 2^{p-1}(2^p+1) - 1$ $(p=5)$ *sets such that the infinite number in*

any one set are equivalent and that the f-curves from different sets are not equivalent. Equivalent f-curves have signatures congruent modulo 2. As types of these sets we may take the $\binom{10}{1}$ *of form* $f_0(i)$, *the* $\binom{10}{2}$ *of form* $f_1(i_1i_2)$, *the* $\binom{10}{5}$ *of form* $f_2(i_1i_2i_3i_4i_5)$, *the* $\binom{10}{6}$ *of form* $f_3(i_1^2i_2i_3i_4i_5i_6i_7)$, *and the* $\binom{10}{9}$ *of form* $f_4(i_1^3i_2i_3i_4i_5i_6i_7i_8i_9)$.

Since all f-curves with signatures congruent modulo 2 are equivalent under the conjugates of a Bertini involution, there follows that the subgroup $g(2)$ of $g_{10,2}$ which is congruent modulo 2 to the identity is generated by these conjugates. Now the index of $g(2)$ under $g_{10,2}$ is the order of the finite group of permutations, $g_{10,2}^{(2)}$, of the above 527 sets. The order of this group has been determined in "T. M. Groups." In fact the signatures of the f-curves reduce modulo 2 to the coefficients of the forms b_1, c_2 of the table there given (p. 323 for $\nu=\varkappa=2$). They are permuted like the even characteristics of the theta functions for $p=5$ under the group (p. 337 *loc. cit.*) of order

$$\mu=2^{21}(2^5-1)(2^8-1)(2^6-1)(2^4-1)(2^2-1),$$

which leaves one even theta characteristic unaltered. Now $g(2)$ is simply isomorphic either with $\overline{G}_{10,2}$ or with a subgroup of it. In the first case μ divided by 10! (to account for the mere ordering of the set P_{10}^2) will be the number of sextics projectively distinct from S and including S itself. In the second case this number will be a smaller factor of $\mu/10!$ Now assuming that μ is the proper index of $\overline{G}_{10,2}$ under $G_{10,2}$ then the index μ' of $\overline{G}_{9,2}$ under $G_{9,2}$ (where these new groups are defined precisely as the groups $\overline{G}_{10,2}$ and $G_{10,2}$ except that all Cremona transformations employed are to have p_{10} as an ordinary point*) is $\mu/527$, since p_{10} or $f_0(10)$ is to be unaltered. Then the number of projectively distinct sets P_9^2 which can be obtained by Cremona transformation from the nine nodes of a sextic is $\mu/527$ divided by 9! But according to P. S. II (47) $\mu/9!527=2^8.960$ is precisely this number of sets P_9^2. Hence μ is the index of $\overline{G}_{10,2}$ under $G_{10,2}$ and $\overline{G}_{10,2}$ is generated by the conjugates of the Bertini involution. We have thus completed the proof of (4) and have also proved that

(9) *A rational plane sextic with ten nodes can be transformed by Cremona transformation into precisely* $2^{13}.31.51$ *projectively distinct sextics. Under such transformation these projectively distinct types (with*

* It is proved in P. S. II, § 6, that the generators of $\overline{G}_{9,2}$ are conjugates of Bertini involutions whence all the Cremona transformations with F-points at P^2_9 for which P^2_9 is congruent to itself will leave unaltered the 10-th node of a sextic with nodes at P^2_9.

ordered nodes) are permuted according to the finite group of odd and even theta characteristics for $p=5$, which leaves an even characteristic unaltered. The infinite discontinuous group $\overline{G}_{10,2}$ of Cremona transformations which leaves S unaltered is simply isomorphic with the subgroup $\bar{g}_{10,2}$ of $g_{10,2}$, which is congruent to the identity modulo 2.

§ 2. *The Discriminant Conditions for P_{10}^2.*

In P. S. II, § 8, the set P_7^2 was discussed in connection with the general plane quartic and the sixty-three factors of the discriminant of this quartic arose from the conditions that two points of P_7^2 should coincide, that three should be on a line, and that six should be on a conic. In all these cases an f-curve passes through one more point of the set than is true in general. The conditions might be indicated thus:

$$f_0(i_1i_2)=0 \quad f_1(i_1i_2i_3)=0, \quad \text{and} \quad f_2(i_1i_2i_3i_4i_5i_6)=0.$$

Similarly for the set P_6^2 (P. S. III (1917), § 1), the same conditions give rise to the thirty-six factors of the discriminant of the cubic surface which is mapped from the plane by cubic curves on P_6^2. We shall therefore continue to refer to such conditions as *discriminant conditions for the set*, even though for sets beyond P_8^2 the word discriminant does not have its usual meaning.

For a general set P_{10}^2 the number of these discriminant conditions is infinite, but they all arise from any one—say $f_0(12)=0$—by Cremona transformation. On the other hand when P_{10}^2 is the special set of ten nodes of a sextic S and therefore subject to three conditions, the existence of one discriminant condition—a fourth condition on P_{10}^2—taken together with the three conditions already implied by the existence of S entails the existence of infinitely many discriminant conditions. For example, reverting to the table (5) of § 1, let us begin with the condition $f_0(1, 9)=0$ which indicates the existence of a tacnode due to the coincidence in some direction of the nodes p_1, p_9. Transforming this by A_{123} we get the condition $f_1(239)=0$ which expresses that the nodes p_2, p_3, p_9 are on a line. Transforming this by A_{145} we get the condition $f_2(123459)=0$, and this, transformed by A_{678} gives rise to $f_4(123456^27^28^29)=0$. But according to the equivalence (1^0) there is a transformation of $\overline{G}_{10,2}$ which leaves the nodes of S unaltered and transforms $f_4(123456^27^28^2)$ into $f_2(12345)$. Therefore if $f_4(123456^27^28^29)=0$ then also $f_2(123459)=0$. Proceeding thus we find that the equivalences of f-curves under $\overline{G}_{10,2}$ imply the identity of corresponding discriminant conditions and we can prove at once by the foregoing methods the theorem:

(10) *The number of discriminant conditions—infinite for the general point set P_{10}^2—is finite for the P_{10}^2 of nodes of S, a set subject to three conditions and containing nine absolute constants. Any two discriminant conditions whose signatures are congruent modulo 2 impose the same* FOURTH *condition on the ten nodes. The $\binom{10}{2}$ conditions of type $f_0(i_1i_2)=0$, the $\binom{10}{3}$ of type $f_1(i_1i_2i_3)=0$, the $\binom{10}{6}$ of type $f_2(i_1i_2i_3i_4i_5i_6)=0$, the $\binom{10}{7}$ of type $f_3(i_1^2i_2i_3i_4i_5i_6i_7i_8)=0$, and the $\binom{10}{10}$ of type $f_4(i_1^3i_2i_3i_4i_5i_6i_7i_8i_9i_{10})=0$, $496=2^{p-1}(2^p-1)$ $(p=5)$ in all, exhaust the number of independent discriminant conditions. The members of this finite set of conditions are permuted under Cremona transformation like the odd theta characteristics under the group of* § 1 (9).

In fact these conditions correspond to the forms b_2, c_3 of the table cited above from T. M. Groups.

From any equivalence there will follow a theorem concerning a special sextic S. Thus from (4^0) and (5^0) of § 1 we have

(11) *If there exists a cubic curve on seven nodes of S with a double point at one of the three remaining nodes (one condition on S) then there will exist a cubic curve on the same seven nodes and with a double point at* ANY *one of the three remaining nodes.*

(12) *If there exists a quartic curve with triple point at one node of S and on the other nodes, then there will exist a quartic with a triple point at any one node and on the other nodes.*

Part of the content of theorem (10) has been stated by Miss Hilda Hudson,* and her method (Section 4, *loc. cit.*) of proving the equivalence of discriminant conditions is interesting. Unfortunately much of this paper is colored by the false assumption that the rational sextic with which she begins, and which has six nodes on a conic is a general rational sextic with nine absolute constants. Miss Hudson uses a space sextic of genus 4—the complete intersection of a quadric and a cubic surface—and assigns to it four actual nodes by making the cubic touch the quadric at four points, and projects it from an arbitrary point of space. Now if λ, μ are the binary parameters of the generators on the quadric, the sextic of genus 4 has the equation $(a\lambda)^3(b\mu)^3=0$ with fifteen constants. Of these six can be removed by projectivities on λ, μ whence the curve has nine absolute constants. These are in

* "The Cremona Transformations of a Certain Plane Sextic," *Proceedings of the London Mathematical Society*, Ser. 2, Vol. XV (1916–17), p. 385.

fact its Riemannian moduli since the curve is normal. The four node requirement reduces the number of constants to 5, and projection from an arbitrary point introduces three more, so that the resulting rational sextic has but eight absolute constants and is subject to the further condition that six nodes are on a conic—a well-known condition on the nodes of any projection of the general space sextic of genus 4. The general rational plane sextic should be obtained as the projection of a general rational space sextic, and the latter sextic does not lie on a quadric.

In the same volume of the *Proceedings* Mr. J. Hodgkinson * shows that there can be *at most* thirty rational sextics with nine properly assigned nodes. As a matter of fact this number is exactly twelve.

In view of these misconceptions it may be worth while to develop in some detail the conditions on the nodes of a rational sextic.† Let then $p_1, \ldots, p_8$ be eight general points of the plane with eight absolute constants. They are the base points of a pencil of cubics $C_\lambda = \lambda_1 C_1 + \lambda_2 C_2$ which meet again in a 9-th point P. This is of course a general pencil of cubics, and all of its members are nondegenerate and all are elliptic except for the twelve nodal cubics of the pencil with nodes at $D_1, \ldots, D_{12}$. The net of sextics, $\mu_1 C_1^2 + \mu_2 C_1 C_2 + \mu_3 C_2^2$, has nodes at $p_1, \ldots, p_8$ and is merely the aggregate of pairs of the pencil C_λ. Other sextics with these nodes exist. Such for example is the degenerate sextic $f_1(12) \cdot f_5(123^2 \ldots 8^2)$ whose factors are known to exist and to be unique. Moreover, this sextic is not found in the above net since it is not a pair of cubics of the pencil C_λ. Let then Σ be any sextic, not included in the net, which has double points at $p_1, \ldots, p_8$. The web of sextics

$$\mu_1 C_1^2 + \mu_2 C_1 C_2 + \mu_3 C_2^2 + \mu_4 \Sigma \tag{13}$$

contains *all* sextics with nodes at $p_1, \ldots, p_8$. For if another sextic Σ' not contained in the system (13) should exist, the system of ∞^4 sextics obtained by adjoining Σ' would cut the line $f_1(12)$ in ∞^2 variable pairs and a pencil would have the fixed factor $f_1(12)$ and the variable factor $f_5(123^2 \ldots 8^2)$ contrary to the fact that this quintic is unique.

All the sextics of the web (13) on a point x pass through a second point y, and x, y are partners in the Bertini involution B.‡ In fact, if C_1 is the cubic

* "The Nodal Points of a Plane Sextic," *loc. cit.*, p. 343.

† Cf. E. C. Valentiner, *Tidsskrift for Math.*, Ser. 4, Vol. V (1881), p. 88, and G. Halphen, *M. S. F. Bull.*, Vol. X (1882), p. 162.

‡ Cf. V. Snyder, "The Involutorial Birational Transformation of the Plane of Order Seventeen," this Journal, Vol. XXXIII (1911), p. 327.

of the pencil C_λ on x, then C_1^2 and C_1C_2 are two independent sextics on x; let $\bar{\bar{\Sigma}}$ be a third. These sextics all meet at the intersections of C_1 and $\bar{\bar{\Sigma}}$. Let the elliptic parameters on C_1 of $p_1, \ldots, p_8$ be $u_1, \ldots, u_8$ (with $u+u'+u''\equiv 0$ as the linear condition) and let u_x, u_y be those of x, y. Then

$$2(u_1+\ldots+u_8)+u_x+u_y\equiv 0.$$

Hence x, y are on a line with the point $u=2(u_1+\ldots+u_8)$ and this is the tangential point of the four points $u=-(u_1+\ldots+u_8)+\frac{\omega}{2}$. If $\frac{\omega}{2}$ is the zero half-period, this is the 9-th base point P; if $\frac{\omega}{2}$ is a proper half-period we we may call the points the *three half-period points on* C_1. Hence a construction for B is as follows: At P, a base point of the pencil C_λ, draw a tangent to the cubic C_λ to meet the cubic C_λ at P_λ, and from P_λ project the cubic into itself to obtain the pairs x, y of B. One easily verifies that the locus of P_λ is a rational quartic on $p_1, \ldots, p_8$ with triple point at P whose tangents are those of cubics with flexes at P. The construction for y is indeterminate only when x is at p_1, or p_2, or $\ldots$, or p_8. The sextic S_{p_1} with triple point at p_1 and nodes at $p_2, \ldots, p_8$ exists and is unique (as is proved at once by reducing its order by a quadratic transformation), and, if x is at any point of S_{p_1}, y is at p_1. Hence B has eight six-fold F-points p_i with corresponding f-curve S_{p_i} and is of order 17. Evidently every sextic (13) and every cubic C_λ is a fixed curve of B.

We are interested primarily in the fixed points of B. These occur at the point P and at the three half-period points on C_λ. The latter run over a locus N which has triple points at p_i with the same tangents as S_{p_i} since these three directions at p_i are self corresponding. Also N is of order 9 since a cubic C_λ meets it in three points outside the eight points p_i. The fixed point P and the fixed point p_9—a general point on N—are of different kinds. P is a fixed point with fixed directions, *i. e.*, a curve K on P is transformed by B into a curve K' on P which touches K. This follows from the fact that P is a fixed point on *every* cubic of the pencil C_λ. On the other hand p_9 is a fixed point on but one cubic C_9 of the pencil C_λ and arises from the coincidence at p_9 in the direction of the tangent T_9 to C_9 at p_9 of a copair x, y of B. Hence this is one fixed direction on p_9, and another is the direction T_{Np_9} of N at p_9, *i. e.*, the direction to a neighboring fixed point. Any curve K on p_9 is transformed by B into a curve K' on p_9 such that the tangents to K and K' at p_9 are harmonic to T_9 and T_{Np_9}.

Every point x of the plane is a double point of at least one sextic of the web, namely of the squared cubic, C_1^2, on it. If x is a double point of a second sextic $\bar{\Sigma}$, and therefore of a pencil, then the net determined by C_1^2, C_1C_2, and $\bar{\Sigma}$ on x have their remaining intersection y at x, which may be at P if $\bar{\Sigma}$ is $C_\lambda^2 (\lambda \neq 1)$, but otherwise is a point p_9 on N. Conversely the net of sextics on p_9 being fixed curves have as a common direction that of T_9 which belongs to the coincident pair, and therefore a pencil of the net will have a node at p_9 with nodal tangents harmonic to T_9 and to T_{Np_9}. The pencil contains one cuspidal sextic with tangent T_{Np_9} and one squared cubic C_9^2 with tangent T_9. Hence, disregarding nodes and cusps due to the sextics C_λ^2, and disregarding also the point P, we see that N is the locus of nodes of sextics of the web (13); or also the locus of cusps of sextics of the web; or as an envelope is the locus of cusp tangents; or finally is that 9-ic with triple points at $p_1, \ldots, p_8$ and on $D_1, \ldots, D_{12}$. For a double point of a cubic C_λ is projected into itself from a point of C_λ. An equation of N is the Jacobian, $J(C_1, C_2, \Sigma)=0$.

The curve N is of genus 4 and its canonical series g_6^3 is cut out by the web of adjoints (13). The series cut out by the pencil C_λ, a g_3^1, has for residue with respect to g_6^3 the same g_3^1. Thus N differs from the general curve of genus 4 in that the two series, g_3^1, cut out on the norm curve by the two sets of generators of the quadric on the norm curve have coincided, *i. e.*, its canonical adjoints (13) map N into a space sextic cut out on a quadric cone by a cubic surface. Since the quadric is a cone, N has but eight moduli, the absolute constants of $p_1, \ldots, p_8$. A tangent plane of the quadric cone does not count as a tritangent plane of the sextic since it is rather a reunion of a set of g_3^1 and a set of $g_3^{1'}$. The 120 tritangent planes arise from the 120 degenerate sextics, $\binom{8}{1}$ of type $f_0(1)\cdot S_{p_1}$, $\binom{8}{2}$ of type $f_1(12)\cdot f_5(123^2\ldots 8^2)$, $\binom{8}{5}$ of type $f_2(12345)f_4(123456^27^28^2)$, and $\binom{8}{6}$ of type $f_3(1^2234567)\cdot f_3(2345678^2)$. Since a g_n^r has $(r+1)(n+rp-r)$ $(r+1)$-fold points, g_3^1 has twelve double points which are at $D_1, \ldots, D_{12}$. If p_9 is a general point on N there is as we have seen, a pencil of sextics with a node at p_9. This pencil cuts N in a g_4^1 with fourteen double points. Two of these double points arise from the two further intersections of the squared cubic C_9^2 on p_9. The remaining twelve are points p_{10} cut out by sextics with a node at p_{10} since all sextics on p_{10} with a simple point at p_{10} touch the cubic C_{10} at p_{10} and not N. Hence in a pencil of sextics with nodes at $p_1, \ldots, p_9$ there are precisely twelve rational sextics. In part this conclusion could be drawn as follows: If p_{10} is the 10-th node of a sextic with nodes at $p_1, \ldots, p_9$ then p_{10} lies both on N and on the 9-ic N'

formed like N with triple points at $p_1, \ldots, p_7, p_9$. Then N and N' meet in 7×9 points at $p_1, \ldots, p_7$ and in 2×3 points at p_8, p_9, whence p_{10} is one of the twelve remaining intersections. Thus there are *at most* twelve positions of p_{10}. It appears therefore that the three conditions that N be on p_9 and p_{10} and that N' be on p_{10} are necessary and sufficient conditions that P_{10}^2 be the nodes of a rational sextic.

The relation between p_9 and p_{10} gives rise to a symmetrical (12, 12) correspondence, T, on N. The valence of T is 3. For if C_3 is a set of the g_3^1, and C_2 the residue of that set on p_9, if K is a canonical set in g_6^3, and G a set of the g_4^1 considered above, and if S_{12} is the set of twelve positions of p_{10} when p_9 is given, then $S_{12}+C_2$ is the set of fourteen double points of the g_4^1. Hence $K+2G\equiv S_{12}+C_2$,* where now the equivalence refers to point groups on N. But $G+2p_9\equiv K$, and $C_2+p_9\equiv C_3$, and $2C_3\equiv K$ whence $S_{12}+3p_9\equiv 2K+C_8$. Hence if p_9' is any other point on N and S_{12}' its set of twelve additional nodes $S_{12}+3p_9\equiv S_{12}'+3p_9'$, or T has the valence $\gamma=3$. Then according to the well-known formula $\alpha+\beta+2p\gamma$, T has $12+12+24=48$ coincidences. These arise from those positions of p_9 where a rational sextic of the web has a tacnode, but also from the twelve points $D_1, \ldots, D_{12}$. For if C_λ has a node at D on N, then C_λ^2 meets N four times at D. Of this $4D$, the set $2D$ is eliminated in forming g_4^1, but $2D$ is left and D is a double point of g_4^1. Thus D belongs to the set S_{12} which corresponds to D in T and is therefore a coincidence. Hence

(14) *There are thirty-six sextics with eight given nodes which have an additional tacnode.*

Thus a sextic with a tacnode has only eight absolute constants. Miss Hudson's theorem that any rational sextic S for which a discriminant condition vanishes can be transformed into a sextic with a tacnode shows that S could have only eight absolute constants. For the tacnodal sextic can be transformed back into S by a series of quadratic involutions each with F-points and one fixed point at nodes of the sextic, and by a subsequent projectivity—a process which can introduce no new absolute constants.

The discriminant conditions furnish irrational invariants of the general sextic S. Symmetric combinations of those which lie within one of the five types of Theorem (10) furnish rational projective invariants of S. Symmetric combinations of the whole set of 496 furnish invariants of S under Cremona transformation of S into S'.

---

* Severi, "Lezioni di Geometria Algebrica," p. 160.

§ 3. *The Group $\overline{G}_{10,2}$ of S.*

Since $\overline{G}_{10,2}$ is the group of all Cremona transformations which transform S into itself, the elements of $\overline{G}_{10,2}$ will either leave every point on S unaltered or transform the points of S among themselves according to a transformation on the parameter t of S of the form

$$t' = \frac{at+b}{ct+d}. \tag{15}$$

The group $\gamma_{10,2}$ of transformations (15) thus induced by $\overline{G}_{10,2}$ upon S will be simply isomorphic with $\overline{G}_{10,2}$ if the group $\overline{\overline{G}}_{10,2}$ of Cremona transformations for which every point of S is fixed is merely the identity. Otherwise $\gamma_{10,2}$ is the factor group of $\overline{\overline{G}}_{10,2}$ under $\overline{G}_{10,2}$.

The $\overline{G}_{10,2}$ is generated by the conjugates of the Bertini involution under $G_{10,2}$. If B is the involution with F-points at the nodes $p_1, \ldots, p_8$ of S, then we have just seen that B leaves the points p_9 and p_{10} unaltered and interchanges the two branches of S at each of these nodes. Hence if t_9, t_9' and t_{10}, t_{10}' are the pairs of nodal parameters, the transformation (15) induced by B interchanges the parameters in each pair and is the involution whose fixed points are the Jacobian of the nodal pairs. These fixed points are cut out on S by the curve N outside of P_{10}^2.

Two f-curves may meet at an F-point say p_i in P_{10}^2, but ordinarily they pass through p_i with different tangents, *i. e.*, they have at p_i different points in common with the f-curve, $f_0(i)$, which is made up of directions at p_i. We say then they have no *proper* intersection at p_i. Two f-curves may be selected so that they have any number of proper intersections. For as the order of the transformations of $G_{10,2}$ increases, the multiplicity of the f-curves of the transformations at F-points also increases so that the number of proper intersections of these f-curves and $f_0(i)$ increases without limit. Any two f-curves without proper intersections are conjugate under $G_{10,2}$. For the first can be transformed into $f_0(10)$ by an operation of $G_{10,2}$ which at the same time transforms the second into an f-curve on P_9^2; and this finally by an operation of $G_{9,2}$ which $f_0(10)$ unaltered can be transformed into $f_0(9)$. Also since every f-curve has precisely two proper intersections with S we have the theorem:

(16) *The group $\gamma_{10,2}$ of transformations* (15) *on S is generated by a conjugate set of involutions each determined by a pair of fixed points which is the Jacobian of the pairs of proper intersections with S of any two f-curves which have no proper intersections with each other.*

One may show in the same way that if ten f-curves are such that no two have proper intersections at P_{10}^2 they define a Cremona transformation of $G_{10,2}$. In fact the signatures of the f-curves furnish the columns of the matrix of L in (1).

If we transform the involution B by L the f-curves $f_0(9)$ and $f_0(10)$ become $f_{r_9}(1^{\alpha_{19}}\dots . 10^{\alpha_{109}})$ and $f_{r_{10}}(1^{\alpha_{110}}\dots . 10^{\alpha_{1010}})$. It merely requires a multiplication of three determinants to form the transform $L^{-1}BL$, and after evident reductions we find that the transformed form has coefficients

$$(17)\qquad \begin{cases} m'=17+12(r_9+r_{10})+4r_9r_{10}, \\ r'_i=s'_i=6+2(r_9+r_{10})+6(\alpha_{i9}+\alpha_{i10})+2(r_9\alpha_{i10}+r_{10}\alpha_{i9}), \\ (j\neq i)\,\alpha'_{ji}=2+2(\alpha_{j9}+\alpha_{j10}+\alpha_{i9}+\alpha_{i10})+2(\alpha_{i9}\alpha_{j10}+\alpha_{j9}\alpha_{i10}), \\ \alpha_{ii}=3+4(\alpha_{i9}+\alpha_{i10})+4\alpha_{i9}\alpha_{i10}. \end{cases}$$

(18) *If* $f_{r_9}(1^{\alpha_{19}}\dots . 10^{\alpha_{109}})$ *and* $f_{r_{10}}(1^{\alpha_{110}}\dots . 10^{\alpha_{1010}})$ *are two f-curves without proper intersections the conjugate of the Bertini involution determined as in* (16) *by the two when regarded as an element L of* $g_{10,2}$ *has the coefficients* (17).

The question as to whether $\overline{\overline{G}}_{10,2}$ contains elements other than the identity is related to the question as to whether the two proper intersections of distinct f-curves with S can coincide. For if $C\neq 1$ is an element of $\overline{\overline{G}}_{10,2}$ and leaves every point of S unaltered, it leaves the two directions of S at p_i unaltered, whence the f-curve which corresponds to p_i under C must pass through p_i with these two directions (and in general others). Thus this f-curve and $f_0(p_i)$ have the same pair of proper intersections with S. I am inclined to think that distinct f-curves meet S in distinct pairs, but have no proof that this is true.

PART II.

THE TEN NODES OF THE SYMMETROID.

§ 4. *The Dilation of a Regular Cremona Group.*

A *regular* Cremona transformation in S_k is by definition (P. S. II, § 4) any product of involutions of the type $y'_iy_i=C_i$ $(i=1, 2, \dots, k+1)$ where the products are formed with the $(k+1)$ F-points within a given point set as described in the introduction. The regular group $G_{n,k}$ attached to the point set P_n^k, transforms spreads of order x_0 and multiplicities $x_1, \dots, x_n$ at P_n^k

according to the group $g_{n,k}$ of linear transformations L with coefficients (P. S. II, § 5 (23))

$$(19)\qquad \begin{pmatrix} (k-1)\mu+1-\rho_1 & -\rho_2 & \vdots & -\rho_n \\ (k-1)\sigma_1 & -\alpha_{11}-\alpha_{12} & \vdots & -\alpha_{1n} \\ (k-1)\sigma_2 & -\alpha_{21}-\alpha_{22} & \vdots & -\alpha_{2n} \\ \cdots\cdots & \cdots\cdots & \vdots & \cdots \\ (k-1)\sigma_n & -\alpha_{n1}-\alpha_{n2} & \vdots & -\alpha_{nn} \end{pmatrix}.$$

This group $g_{n,k}$ is generated by the permutation $g_{n!}$ of the n variables and the involution $A_{1,2,\ldots\ldots,k+1}$ whose coefficients are (P. S. II, § 5)

$$(20)\qquad \begin{pmatrix} k & -1 & -1 & \vdots & -1 & 0 & \vdots \\ k-1 & 0 & -1 & \vdots & -1 & 0 & \vdots \\ k-1 & -1 & 0 & \vdots & -1 & 0 & \vdots \\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \vdots \\ k-1 & -1 & -1 & \vdots & 0 & 0 & \vdots \\ 0 & 0 & 0 & \vdots & 0 & 1 & \vdots \\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \vdots \end{pmatrix}.$$

Suppose then that the general element of $G_{n,k}$ has been obtained by forming a proper sequence Π of the products from $g_{n!}$ and $A_{1,\ldots\ldots,k+1}$. Consider a set of $n+l$ points in an S_{k+l}, *i. e.*, a set P_{n+l}^{k+l}. In this space separate a set of l of the points P_{n+l}^{k+l} (call these for the moment the *fixed* F-points) and order the remaining n points of P_{n+l}^{k+l} with respect to the points of P_n^k. Then in S_{k+l} form a product Π' of elements from $g_{(n+l)!}$ and $A_{1,\ldots\ldots,l,\,l+1,\ldots\ldots,\,l+k+1}$ in such a way that the last n points of P_{n+l}^{k+l} are permuted like the n points of P_n^k under $g_{n!}$, the first l remaining fixed. This requires that always in using an element A the first l of its F-points shall fall at the first l points of the set P_{n+l}^{k+l}. We shall then say that the element Π' of $G_{n+l,\,k+l}$ is the *element* Π *of* $G_{n,k}$ *dilated into* S_{k+l}. The element of $g_{n+l,\,k+l}$ which corresponds to the element Π' dilated from (19) has coefficients

$$(21)\qquad \begin{pmatrix} (k+l-1)\mu+1 & -\mu & -\mu & \vdots & -\mu & -\rho_1 & \cdots & -\rho_n \\ (k+l-1)\mu & -\mu+1 & -\mu & \vdots & -\mu & -\rho_1 & \cdots & -\rho_n \\ (k+l-1)\mu & -\mu & -\mu+1 & \vdots & -\mu & -\rho_1 & \cdots & -\rho_n \\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots \\ (k+l-1)\mu & -\mu & -\mu & \vdots & -\mu+1 & -\rho_1 & \cdots & -\rho_n \\ (k+l-1)\sigma_1 & -\sigma_1 & -\sigma_1 & \vdots & -\sigma_1 & -\alpha_{11} & \cdots & -\alpha_{1n} \\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots \\ (k+l-1)\sigma_n & -\sigma_n & -\sigma_n & \vdots & -\sigma_n & -\alpha_{n1} & \cdots & -\alpha_{nn} \end{pmatrix}.$$

In order to prove this we have only to show that the general element (19) multiplied by $A_{1,\ldots,k+1}$ when dilated according to the rule which is evident in (21) is the same as the dilated element (21) multiplied by $A_{1,\ldots,l+1,\ldots,l+k+1}$. We shall omit the verification which depends merely on determinant multiplication. Hence

(22) *The regular Cremona group attached to a set P_n^k in S_k when dilated into S_{k+l} furnishes a subgroup of the regular Cremona group in S_{k+l} attached to the set P_{n+l}^{k+l} which is simply isomorphic with the original group. The dilated group permutes the S_l's in S_{k+l} upon the l fixed F-points just as the original group permutes the points of S_k.*

In fact if the S_l's be cut by an S_k, which does not cut their common S_{l-1}, the original group appears in this S_k.

The following extension of P. S. II, § 4 (17) is now evident.

(23) *The group $G_{n,k}$ contains subgroups simply isomorphic with $G_{n',k'}$ whenever $n' \leqq n$ and $k' \leqq k$.*

We shall have occasion to use the dilations into S_3 of the Bertini involution, and of the Geiser involution in S_2 with triple F-points at $p_2, \ldots, p_8$. The matrices of these dilated transformations are, respectively,

$$(24)\quad \begin{pmatrix} 33 & -16 & -6 & -6 & \ldots & -6 \\ 32 & -15 & -6 & -6 & \ldots & -6 \\ 12 & -6 & 3 & -2 & \ldots & -2 \\ 12 & -6 & -2 & 3 & \ldots & -2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 12 & -6 & -2 & -2 & \ldots & 3 \end{pmatrix}, \quad \begin{pmatrix} 15 & -7 & -3 & -3 & \ldots & -3 \\ 14 & -6 & -3 & -3 & \ldots & -3 \\ 6 & -3 & -2 & -1 & \ldots & -1 \\ 6 & -3 & -1 & -2 & \ldots & -1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 6 & -3 & -1 & -1 & \ldots & -2 \end{pmatrix}.$$

§ 5. *The Transforms of the Symmetroid by Regular Cremona Transformation.*

The symmetroid Σ is the quartic surface obtained by equating to zero a symmetric determinant of order 4 whose elements are linear forms. The ten points at which the first minors all vanish form the set P_{10}^3 of nodes of Σ. The enveloping cone of Σ from one of the nodes breaks up into two cones of the third order which meet in the nine lines to the other nodes. If this property appears at one node of a ten-nodal quartic surface, the surface is a symmetroid.

Let us call a set of eight points in space $p_1, \ldots, p_8$ a *half-period set* if on the elliptic quartic through the eight, the parameters satisfy the condition $u_1+\ldots+u_8\equiv\omega/2$, where $\omega/2$ is not the zero half-period when $v_1+\ldots+v_4\equiv 0$ is the coplanar condition. Let us further call $8+k$ points a *half-period set* if every set of eight in the set of $8+k$ points is itself a half-period set. Then a further property of Σ is that its set of nodes P_{10}^3 is a half-period set.*

From the property of the enveloping cone there follows:

(25) *If nine nodes of a symmetroid are given, the tenth is uniquely determined.*

(26) *A symmetroid is transformed by regular Cremona transformations with $\rho \overline{<} 10$ F-points at P_{10}^3 into a symmetroid Σ' whose nodes $P_{10}^{3\prime}$ are congruent to P_{10}^3.*

For first if $p_1, \ldots, p_9$ are given, the line $\overline{p_1p_{10}}$ is determined as the 9-th base line of a pencil of cubic cones on the eight lines $\overline{p_1p_2}, \ldots, \overline{p_1p_{10}}$. Similarly the line $\overline{p_2p_{10}}$ is determined and thereby also the node p_{10}. Secondly a cubic transformation A_{1234} of the type $x_i'x_i=C_i$ $(i=1, \ldots, 4)$ with F-points at $p_1, \ldots, p_4$ transforms Σ into a quartic surface Σ' with nodes at a congruent set Q_{10}^4. Now A_{1234} is the dilation of a ternary quadratic transformation A_{234} which sends nine base points of a pencil of cubics on $p_2p_3p_4$ into a congruent set with a similar base point property whence A_{1234} has the same effect on the nine base lines through p_1, and Σ' is also a symmetroid. Moreover, any regular transformation of the sort described in (26) is a product of such cubic transformations.

It is our primary purpose to show that Σ can be transformed by such regular transformation into only a finite number of projectively distinct symmetroids, or since

(27) *There is but one symmetroid with given nodes*, that from the set P_{10}^3 of nodes of Σ only a finite number of projectively distinct congruent sets Q_{10}^3 can be derived.

In general there is an infinite number of sets Q_{10}^3 congruent to but projectively distinct from P_{10}^3 (P. S. II (14), (18)), and these arise from P_{10}^3 by the operations of the group $G_{10,3}$. If for the set P_{10}^3 of nodes of Σ this number is finite, there must be a subgroup $\overline{G}_{10,3}$ of $G_{10,3}$, which transforms Σ into itself, of finite index under $G_{10,3}$. We shall find that an important subgroup

* Cayley, *Coll. Math. Pap.*, Vol. VII, p. 304; Vol. VIII, p. 25.

$G(2)$ of $\bar{G}_{10,3}$ is generated by the conjugates under $G_{10,3}$ of two types of involutions, namely, the "Kantor involution"* and the dilated Bertini involution. In $g_{10,3}$ there are the isomorphic subgroups $\bar{g}_{10,3}$ and $g(2)$.

The Kantor involution K is that cut out on elliptic quartic curves on $p_1, \ldots, p_7$ by quartic surfaces with nodes at $p_1, \ldots, p_7$. It has for fixed points the 8-th node of such surfaces; and these are the 8-th base point P of the net of quadrics on $p_1, \ldots, p_7$—an isolated fixed point with fixed directions—and the locus of the point p_8 which forms with $p_1, \ldots, p_7$ a half-period set—the Cayley dianome sextic surface. Hence p_8, p_9, p_{10}, the further nodes of Σ, are fixed points of K, and Σ is unaltered by K. This involution is the analog in space of the Bertini involution in the plane.

In order to show that the dilated Bertini involution also leaves Σ unaltered, two lemmas are useful.

(28) *The dilation from p_1 of the Geiser involution with F-points $p_2, \ldots, p_8$ in S_2 is, in S_3, the transformation* (24) *whose two sets of F-points $p_1, \ldots, p_8$ and $q_1, \ldots, q_8$ are projective only when they are half-period sets. If the two sets are thus restricted and coincide in the identical order, the dilated transformation is involutory.*

For the dilation of this involution is found listed in P. S. II, p. 376, as $C(\nu)\,(\nu=-1)$. It is shown there that $C(-1)C(0)=D_1$ or $C(-1)=D_1C(0)$ where $C(0)$ is the Kantor involution determined by $p_2, \ldots, p_8$. It is clear from the parametric equations of D_1 (*loc. cit.*) that its two sets of F-points are projective if they are half-period sets. In this case p_1 is a fixed point of $C(0)$ and the two sets of F-points of $C(-1)$ are projective. If for $C(-1)$, P_8^3 and Q_8^3 coincide then $p_1, \ldots, p_8$ are ordinary points of $[C(-1)]^2$ and $C(-1)$ is involutory.

(29) *The dilation from p_1 of the Bertini involution with F-points $p_2, \ldots, p_9$ in S_2 is, in S_3, the transformation* (24) *whose two sets of F-points $p_1, \ldots, p_9$; $q_1, \ldots, q_9$ are projective only when these sets are half-period sets. If they are thus restricted and coincide in the identical order, the dilated transformation is involutory.*

For in P. S. II, p. 353, the Bertini involution (E_{17}) is expressed as a product of three Geiser involutions (D_8) and from the projectivity of the two

*The Kantor involution appears first in two papers of S. Kantor, "Theorie der periodischen cubischen Transformationen im R_3," this Journal, Vol. XIX (1897), p. 1; and "Theorie der Transformationen im R_3," *Acta Mathematica*, Vol. XXI (1897), p. 1, both of which deal with regular transformations in S_3. A development of the properties of the involution is given by J. R. Conner, "Correspondences Determined by the Bitangents of a Quartic," this Journal, Vol. XXXVIII (1916), p. 155.

sets of seven F-points of the factors, the projectivity of any two corresponding sets of six points from the two sets of eight F-points of the product was derived. Because of the isomorphism between elements in S_2 and their dilations in S_3 the dilated Bertini involution can be expressed as a similar product of three dilated Geiser involutions. Hence by virtue of (28) we can conclude that the pair p_1q_1 and any six further pairs of F-points are projective when $p_1, \ldots, p_9$ is a half-period set. Hence the two sets of F-points of the dilated Bertini involution are projective when one is a half-period set, and if the two sets coincide the square of the transformation is the identity.

We can now proceed with Σ very much as with the sextic S and will state first the analog of Theorem (4), § 1.

(30) *The group, $G(2)$, of regular transformations in S_3, generated by the conjugates of the Kantor and dilated Bertini involutions under $G_{10,3}$, is an invariant subgroup of $\overline{G}_{10,3}$ which leaves Σ unaltered. The isomorphic group, $g(2)$, is that subgroup of $g_{10,3}$ which is congruent to the identity modulo 2.*

Indeed we have already remarked that K leaves Σ unaltered. There follows directly from (29), (25) and (27) that B has the same property. That the conjugates of K, B under $G_{10,3}$ have this property is proved as for the sextic. In order to prove that the involutions generate $G(2)$ we indicate as before by the symbol $\equiv$ equivalence under $G(2)$.

If P_{10}^3 is the set of nodes of Σ it determines a sequence of f-surfaces, the conjugates of the ∞^2 directions about $p_1, \ldots, p_{10}$ under the operations of $G_{10,3}$. We construct like the Table (5) for the sextic the Table (31), discarding as new types those f-surfaces which are equivalent under $G(2)$ to types found earlier. Non-equivalent new types are starred as they occur.

In order to prove the equivalences listed in the table we shall prove first that the following list is valid.

(12°) $$f_3(1^2 2 5^3 678910) \equiv f_3(1^2 2^3 5678910).$$

(13°) $$f_3(12^3 5^2 678910) \equiv f_3(12^3 3^2 678910).$$

(14°) $$f_2(12^2 3456) \equiv f_2(134567^2).$$

(15°) $$f_4(1^3 2^4 345678910) \equiv f_4(12^4 3^3 45678910).$$

We begin with the equivalences obtained from K and B,

$$f_0(i) \equiv f_4(i^3 j_1^2 \ldots j_6^2) \equiv f_6(i^3 j_1^6 j_2^2 \ldots j_8^2),$$

and transform them successively by $A_{ij_1j_2j_3}$, $A_{ij_1j_4j_5}$, $A_{ij_1j_6j_7}$, and $A_{ij_1j_8j_9}$ getting

$$\begin{array}{lll} f_1(j_1j_2j_3) & \equiv f_3(i^2j_1j_2j_3j_4^2j_5^2j_6^2) & \equiv f_5(i^2j_1^5j_2j_3j_4^2\ldots\ldots j_8^2), \\ f_2(ij_1^2j_2j_3j_4j_5) & \equiv f_2(ij_2j_3j_4j_5j_6^2) & \equiv f_4(ij_1^4j_2j_3j_4j_5j_6^2j_7^2j_8^2), \\ f_3(i^2j_1^3j_2\ldots\ldots j_7) & \equiv f_3(i^2j_1\ldots\ldots j_5j_6^3j_7) & \equiv f_3(j_1^3j_2\ldots\ldots j_7j_8^2), \\ f_4(i^8j_1^4j_2\ldots\ldots j_9) & \equiv f_6(i^5j_1^4j_2\ldots\ldots j_5j_6^3j_7j_8^3j_9^3) & \equiv f_4(ij_1^4j_2\ldots\ldots j_8j_9^3). \end{array}$$

(31)

Type	operated upon by	becomes	which is	
$f_0(1)$	A_{1234}	$f_1(234)$ *		
	A_{2345}	$f_0(1)$		
$f_1(234)$	A_{1234}	$f_0(1)$		
	A_{1235}	$f_1(234)$		
	A_{1256}	$f_2(12^23456)$ *		
	A_{1567}	$f_3(1^22345^26^27^2)$	$\equiv f_1(234)$	(1^0)
$f_2(12^23456)$	A_{1234}	$f_1(256)$		
	A_{3456}	$f_2(12^23456)$		
	A_{1237}	$f_2(12^23456)$		
	A_{4567}	$f_3(12^234^25^26^27)$	$\equiv f_1(137)$	(1^0)
	A_{1278}	$f_3(1^22^3345678)$ *		
	A_{5678}	$f_4(12^2345^36^37^28^2)$	$\equiv f_2(12^23456)$	(2^0)
	A_{2789}	$f_4(12^434567^28^29^2)$	$\equiv f_2(12^23456)$	(3^0)
	A_{1789}	$f_5(1^42^234567^38^39^3)$	$\equiv f_3(1^234567^389)$	(4^0)
	A_{78910}	$f_6(12^234567^48^49^410^4)$	$\equiv f_2(12^23456)$	(5^0)
$f_3(1^22^3345678)$	A_{1234}	$f_2(12^25678)$		
	A_{2345}	$f_3(1^22^3345678)$		
	A_{1345}	$f_4(1^32^33^24^25^2678)$	$\equiv f_2(123^2678)$	(2^0)
	A_{3456}	$f_5(1^22^33^34^35^36^378)$	$\equiv f_3(1^22^3345678)$	(6^0)
	A_{1289}	$f_3(1^22^3345678)$		
	A_{2789}	$f_4(1^22^434567^28^29)$	$\equiv f_2(1^234569)$	(3^0)
	A_{1789}	$f_5(1^42^334567^38^39^2)$	$\equiv f_3(1^22^3345678)$	(4^0)
	A_{6789}	$f_6(1^22^33456^47^48^49^3)$	$\equiv f_2(1^223459)$	(7^0)
	A_{12910}	$f_4(1^32^43\ldots10)$ *		
	A_{28910}	$f_5(1^22^53\ldots78^39^210^2)$	$\equiv f_3(1^22^3345678)$	(8^0)
	A_{18910}	$f_6(1^52^33\ldots78^49^310^3)$	$\equiv f_4(1^32\ldots78^4910)$	(9^0)
	A_{78910}	$f_7(1^22^334567^58^59^410^4)$	$\equiv f_3(1^22^3345678)$	(10^0)
$f_4(1^32^43\ldots\ldots10)$	A_{1234}	$f_3(1^22^35678910)$		
	A_{2345}	$f_5(1^32^53^24^25^2678910)$	$\equiv f_3(12^33^2678910)$	(8^0)
	A_{1345}	$f_6(1^52^43^34^35^3678910)$	$\equiv f_4(1^32^43\ldots10)$	(9^0)
	A_{3456}	$f_8(1^32^43^54^55^56^578910)$	$\equiv f_4(1^32^43\ldots10)$	(11^0)

In these transforms we find (12^0), (13^0), (14^0), (15^0) as well as (1^0), (2^0) and (9^0). Also (2^0) is transformed by A_{1234} into (6^0), whence (6^0) is valid. Since (8^0) is transformed by A_{1234} into (13^0), (8^0) also is valid. Again (7^0) is

transformed by A_{1267} and the use of (13^0) into (4^0), and (4^0) by A_{1789} into (14^0). Also (3^0) is transformed by A_{1234} into one proved above. The equivalence (11^0) is transformed by A_{1234} into (10^0), and (10^0) by A_{1278} into (5^0). Finally, by using (14^0) we write (5^0) as $f_6(12^2 34567^4 8^4 9^4 10^4) \equiv f_2(134567^2)$ and this is transformed by A_{1789} and the use of (13^0) into (4^0). According to the equivalences derived above from the conjugates of K and B we find that all f-surfaces whose signatures are congruent modulo 2 are equivalent under $G(2)$ which completes the proof of (30).

The factor group of $g(2)$ under $g_{10,8}$ is the group $g_{10,8}^{(2)}$ of transformations L reduced modulo 2. According to the table (T. M. Groups, p. 337, $\kappa=3, \nu=2$) this group has the order $\mu=2^9 . 2^{16}(2^8-1)(2^6-1)(2^4-1)(2^2-1)$. Also μ is the index of $G(2)$ under $G_{10,3}$. There may be elements in $G_{10,3}$ other than those in $G(2)$ which leave Σ unaltered. Consider the μ transforms of Σ under $G_{10,3}$. In these transforms we find that the f-surface $f_0(10)$ is transformed into 2^9 conjugates not equivalent under $G(2)$. These are of the five types listed in the first column of Table (31), there being $\binom{10}{1}$, $\binom{10}{3}$, $\binom{10}{5}$, $\binom{10}{7}$, $\binom{10}{9}$ of the respective types. Hence, under the operations of $G_{10,3}$ for which p_{10} is an ordinary point, we would find only $\mu'=\mu/2^9$ transforms of Σ. Under the latter operations the f-surface $f_0(9)$ is transformed into 2^8-1 conjugates not equivalent under $G(2)$, namely the $\binom{9}{1}$, $\binom{9}{3}$, $\binom{9}{5}$, $\binom{9}{7}$ of the first four types just mentioned. Hence under the operations of $G_{10,3}$ for which both p_9 and p_{10} are ordinary points, we would get only $\mu''=\mu'/(2^8-1)=2^{16}(2^6-1)(2^4-1)(2^2-1)$ transforms of Σ, and these recur in sets of 8! obtained by permutation of $p_1, \ldots, p_8$. Thus we should get only $\mu''/8!=2.2^6.36$ projectively distinct sets of nodes $p_1, \ldots, p_8$. On the other hand we have proved (P. S. II, p. 377 (46)) that when P_8^3 is a half-period set, there are only $2^6.36$ projectively distinct sets congruent in some order to P_8^3.

This indicates the existence of Cremona transformations not in $G(2)$ which have their F-points at $p_1, \ldots, p_8$ alone and which transform Σ into itself. Indeed

(32) *The dilated Geiser involution with F-points at the nodes $p_1, \ldots, p_8$ of Σ transforms Σ into itself and interchanges the nodes p_9 and p_{10}.*

For let us first recall with Rohn * that when the first seven nodes of Σ are given, the other three lie on Cayley's dianome sextic surface with triple points at $p_1, \ldots, p_7$. Having chosen p_8 on this surface, the other two nodes lie on

* K. Rohn, "Die Flächen vierter Ordnung," etc., Jablonowski'schen Preisschrift, Leipzig (1886), §§ 9, 10, 11.

Cayley's dianodal curve of order 18 with planar triple points at the eight nodes. As Rohn remarks, the ninth being chosen, the tenth is uniquely determined if the quartic is to be a symmetroid. This follows immediately from (25). Thus there is on the dianodal curve an involution of pairs of nodes of symmetroids. Now this involution is effected by the Geiser involution dilated from p_1 (and therefore also that the Geiser involution dilated from any other of the eight nodes). For since the eight nodes are a half-period set, the dilated transformation is involutory (28) when its two sets of eight F-points coincide. Moreover, the dilated transformation is regular and transforms symmetroids into symmetroids (26) and therefore leaves the dianodal curve unaltered. If p_9, p'_9 are a copair of the dilated involution on the curve, then from (22) the lines $\overline{p_1p_9}$, $\overline{p_1p'_9}$ form with $\overline{p_1p_2}, \ldots, \overline{p_1p_8}$ the base lines of a pencil of cubic cones. But this property is shared by the lines $\overline{p_1p_9}$ and $\overline{p_1p_{10}}$ when p_9, p_{10} are nodes of the same symmetroid. Hence p'_9 is p_{10} and the theorem is proved.

Consider now the reduced group $g^{(2)}_{10,3}$ of $g_{10,3}$. The dilated Geiser involution reduced modulo 2 is

$$I_{12\ldots 8}I_{910} \text{ or } x'_j=x_j+(x_1+\ldots+x_8) \quad (j=0,1,\ldots,8), \quad x'_9=x_{10}, \quad x'_{10}=x_9$$

in the notation of T. M. Groups.* This is an element T (cf. p. 326, *loc. cit.*) which lies in the invariant g_{2^9} of $g^{(2)}_{10,3}$. If an element of $G_{10,3}$ leaves Σ unaltered, its conjugates have the same property whence those elements of $g^{(2)}_{10,3}$ conjugate to T under $g^{(2)}_{10,3}$, also correspond to elements of $G_{10,3}$ which leave Σ unaltered. Now the factor group of g_{2^9} under $g^{(2)}_{10,3}$ is the *simple* group $G_{NC}(p=4)$ of the odd and even thetas for $p=4$. Hence there are no further elements of $G_{10,3}$ which leave Σ unaltered since any such element reduced modulo 2 would furnish an invariant subgroup of $g^{(2)}_{10,3}$ larger than g_{2^9} whose factor group under $g^{(2)}_{10,3}$ would be the factor group under G_{NC} of an invariant subgroup of G_{NC} greater than the identity. But no such subgroup of G_{NC} exists. Hence the number $\bar{\mu}$ of transforms of Σ under $G_{10,3}$ is the order of G_{NC}, *i. e.*, $\bar{\mu}=2^{16}(2^8-1)(2^6-1)(2^4-1)(2^2-1)$ and allowing for the permutations of the nodes there are only $\bar{\mu}/10!=2^8.51$ projectively distinct Σ's. Hence

(33) *Under regular Cremona transformation a symmetroid Σ can be transformed into precisely $2^8.51$ projectively distinct Σ''s. The subgroup $\overline{G}_{10,3}$ of $G_{10,3}$, which leaves Σ unaltered is generated by the conjugates*

* Cf. particularly the table, p. 337, for $\kappa=3$, $\nu=2$, and also (28) and (29) with references there given.

*under $G_{10,3}$ of the dilated Geiser involution and the Kantor involution.** *The corresponding elements of $\bar{g}_{10,3}$ are characterized arithmetically by the fact that when reduced modulo 2 they yield either the identity or elements which transform the forms b_2, b_4 each into itself or into its paired form.*† *The invariant subgroup $G(2)$ of $G_{10,3}$ for which $g(2)$ is congruent to the identity modulo 2 is generated by the conjugates of the Kantor and dilated Bertini involutions, and has for factor group under $\bar{G}_{10,3}$ an abelian group of involutions of order 2^9. Under $G_{10,3}$ the conjugates of Σ are permuted according to the group of odd and even thetas for $p=4$, the particular types corresponding to the base configurations.* ‡

We may note finally the behavior of the discriminant factors of the set P_{10}^3 of nodes of Σ. Due to the equivaleuces under $G(2)$ listed above we find that all of the discriminant conditions are equivalent to the following sets: $\binom{10}{2}$ of type $f_0(i_1 i_2)$, $\binom{10}{4}$ of type $f_1(i_1 i_2 i_3 i_4)$, $\binom{10}{6}$ of type $f_2(i_1^2 i_2 \ldots . i_7)$, and $\binom{10}{8}$ of type $f_3(i_1^3 i_2^2 i_3 \ldots . i_9)$, or $2(2^8-1)$ in all. But due to the equivalences under elements of $\bar{G}_{10,3}$ not in $G(2)$, these are paired into 2^8-1 pairs, $\binom{10}{2}$ of type $f_0(i_1, i_2)$, $f_3(i_1^2 i_3^3 i_4 \ldots . i_{10})$ and $\binom{10}{4}$ of type $f_1(i_1 i_2 i_3 i_4)$, $f_2(i_1^2 i_5 \ldots . i_{10})$. These two types of equivalence lead to the theorems

(34) *If two nodes of a symmetroid coincide, the cubic cone with vertex at any any one of the remaining nodes and on the ten nodes has a double generator on the double node.*

(35) *If four nodes of a symmetroid are in a plane there is a quadric cone with vertex at any one of the four nodes and on the remaining six nodes.*

When none of the discriminant conditions are satisfied they become irrational invariants of the symmetroid whose behavior under $G_{10,3}$ can be described thus:

(36) *Under regular Cremona transformation the 2^8-1 independent discriminant invariants of Σ are permuted like the points of an $S_{2p-1}(p=4)$ under the group of a null system in S_{2p-1}.*

This striking analogy with the 2^6-1 discriminant invariants of P_7^2 (or the ternary quartic for $p=3$; cf. P. S. II, § 8) is undoubtedly significant.

URBANA, ILLINOIS, *May* 15, 1919.

---

* The dilated Bertini involution can be generated by dilated Geiser involutions.

† The forms b_2 are $x_{i_1}+x_{i_2}$, $x_{i_1}+\ldots+x_{i_6}$, the forms b_4 are $x_{i_1}+\ldots+x_{i_4}$ and $x_{i_1}+\ldots+x_{i_8}$; paired forms taken together make up $x_{i_1}+\ldots+x_{i_{10}}$ $(i_j=1, \ldots, 10)$.

‡ For these configurations cf. a paper of the author on "The Finite Geometry of the Theta Functions," *Trans. Amer. Math. Soc.*, Vol. XIV (1913), p. 271.